A CAT(0) Metric on the Davis Moussong Complex

Donnerstag, 1. Juli 2021 12:47

§0 Review:

- Given a Coxeter System (W,S), there exists a Complex $\Sigma(W,S)$ on which W acts geometrically. Additionally, $\Sigma(W,S)$ is simply connected.

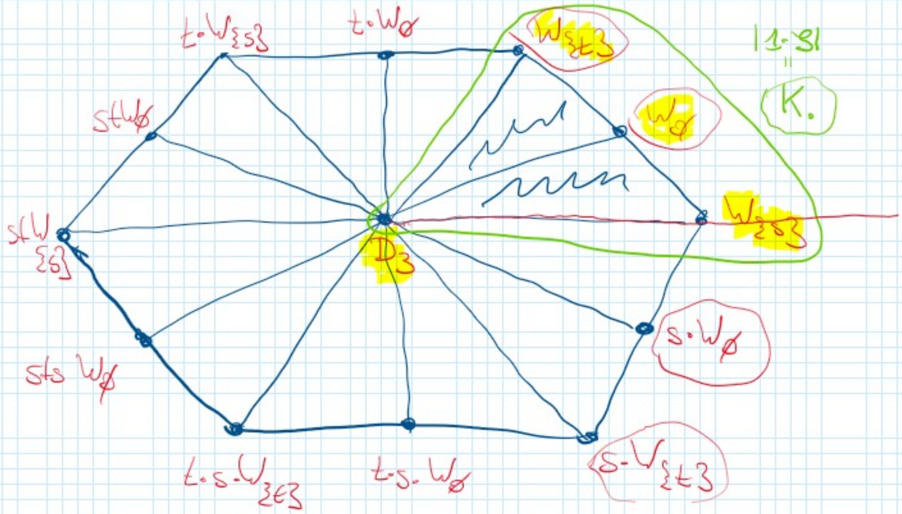
Construction: (W,S) Coxeter System.

$\mathcal{S} := \{ T \subseteq S \mid T \text{ spherical} \}$,

$W\mathcal{S} := \{ w \cdot W_T \mid w \in W,\ T \in \mathcal{S} \}$

$\rightsquigarrow \Sigma(W,S) := |W\mathcal{S}|$ geometric realization

Example: $D_3 := \langle s,t \mid s^2, t^2, (st)^3 \rangle$



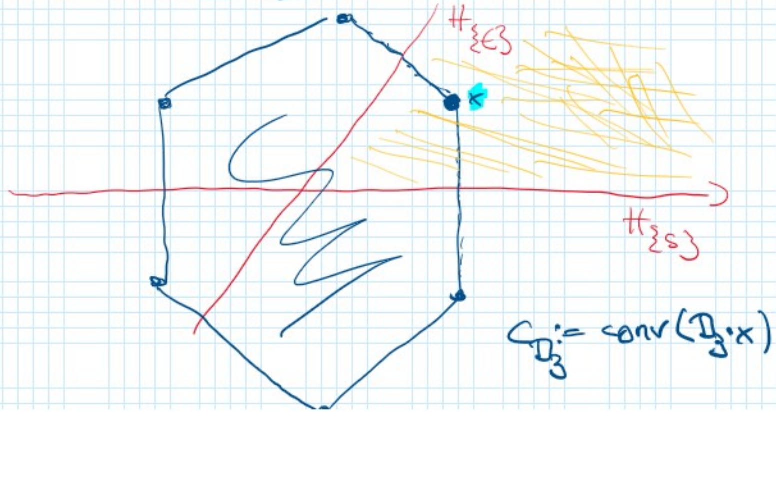
Cell Structure:

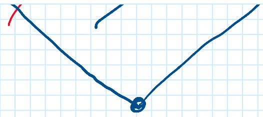
- Coxeter Polytopes.

For $T \subseteq S$ spherical, let C_T denote the associated Coxeter Polytope.

Identify each subcomplex isomorphic to $\Sigma(W_T, T)$ with C_T.

Ex: Coxeter Polytope: D_3

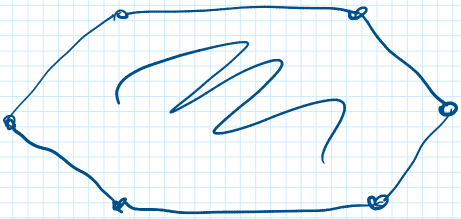




$C_{D_3} := \mathrm{conv}(D_3 \cdot x)$

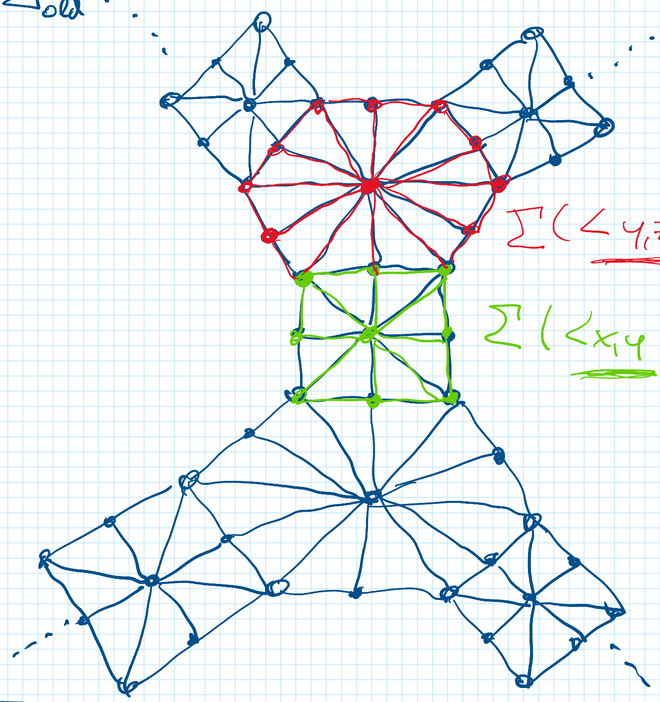
→ From now on consider Σ with this cell structure!

Ex: (1) $\Sigma(D_3, \{s,t\})$



(2) $(U,S) = \langle x,y,z \mid x^2, y^2, z^2, (xy)^2, (yz)^3 \rangle$

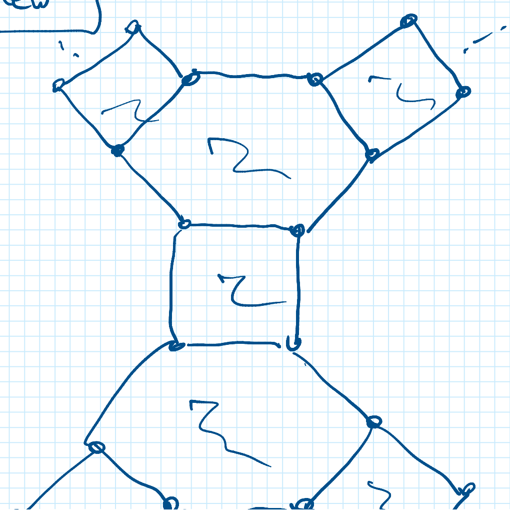
Σ_{old} :

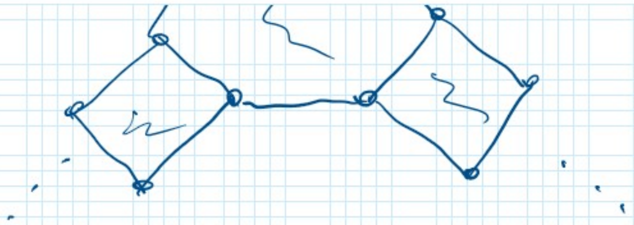


$\Sigma(\langle \underline{y,z} \mid y^2, z^2, (yz)^3 \rangle)$

$\Sigma(\langle \underline{x,y} \mid x^2, y^2, (xy)^2 \rangle)$

$\boxed{\to \Sigma_{new} :}$

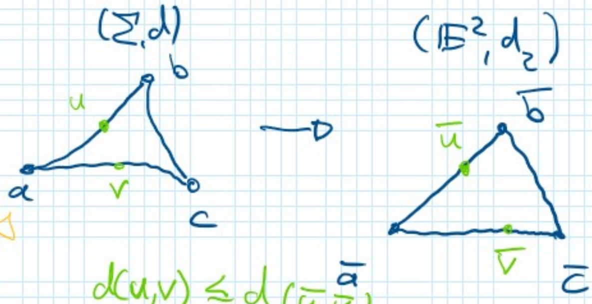




Important Properties:

① Σ is simply connected.

② The Link of each vertex v is isomorphic to the nerve L of (W,S). → Later more details!



Goal:

Define a metric on Σ + show the metric is CAT(0)!

Main "Tools":

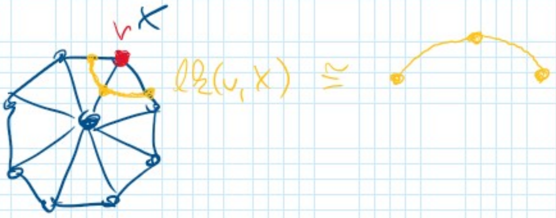
Thm A (Cartan-Hadamard): Let X be a geodesic metric space. Then the following are equivalent:

① X is CAT(0).

② X is locally CAT(0) + X is simply connected.

Thm B: A piecewise Euclidean cell complex is locally CAT(0) iff the link of every vertex is CAT(1). (Gromov / Bridson ?)

Reminder / Idea:



Thm C (Gromov + Moussong)

Let L be a finite spherical complex.

① if all edge lengths are $\boxed{\frac{\pi}{2}}$, then L is CAT(1) $\Leftrightarrow$ L is a flag complex (*)

② if all edge lengths are $\boxed{\geq \frac{\pi}{2}}$, then

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✱ if $\boxed{V = \{v_0, \dots, v_k\}} \subseteq Vert(L)$ is pairwise connected by edges, then V spans a k-simplex.

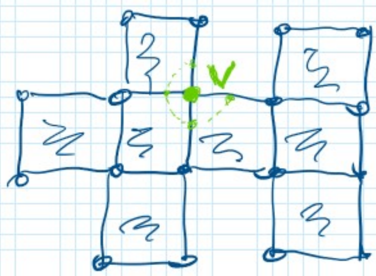
(∗) if $\boxed{V = \{v_0, \dots, v_k\}} \subseteq Vert(L)$ is pairwise connected by edges of length $\boxed{\ell_{ij}}$, such that there exists a k-simplex in $\boxed{S^n}$ with these edge lengths, then V spans a k-Simplex in L.

⇒ combinatoric condition for curvature

⇒ easier to check!!!

Ex:

①



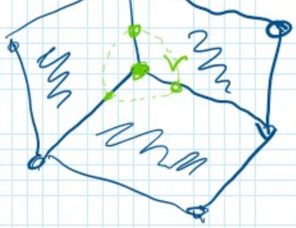
→ $lk(v,X)$

⇒ flag

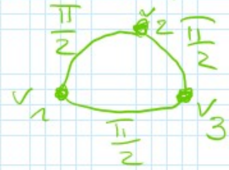
→ check this for all vertices

⇒ $CAT(0)$.

②



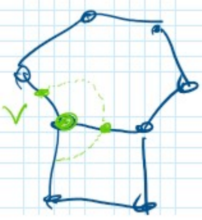
→ $lk(v,X) \cong$



⇒ not flag!

⇒ not $CAT(0)$

③



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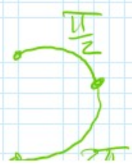


$\Rightarrow$ metric flag complex

$\Rightarrow$ CAT(0).

§1 A metric on Σ.

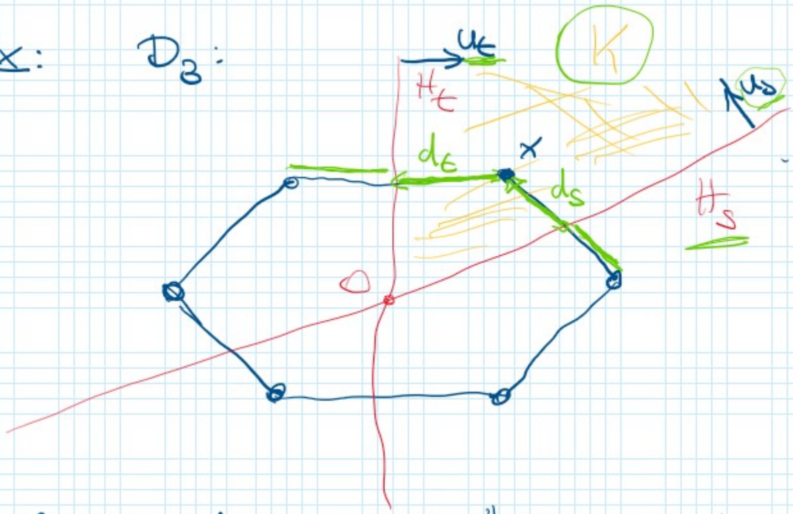
Define a metric on Σ in 2 steps:

① Define a metric on Coxeter polytopes

② Metrics on cell-complexes "gluing metrics".

For step 1:

Ex: D_3:



$\rightarrow$ Given $d_s, d_t > 0$, "there exists one point $x \in K$ with distance d_s to H_s and d_t to H_t".

$\Rightarrow$ Metric on C_{D_3} !

General definition:

- Let (W,S) be a finite Coxeter System and $(d_s)_{s\in S}$ a sequence of positive real numbers. Further let u_i be the unit inward pointing normal vector for the Hyperplane $H_{\{i\}}$.

$\Rightarrow$ Choose x as the unique point in K such that $\langle x, u_s \rangle = d_s$ $\forall s \in S$.

Then define $C_W := \mathrm{conv}(W \cdot x)$. This gives a metric on C_W (!)

- Davis-Moussong complex:

For (V,S) choose a sequence $(d_s)_{s\in S} \in (0,\infty)^S$. and metrize every Coxeter polytope for $T \subseteq S$ spherical using a point x_T and the sequence $(d_t)_{t\in T}$.

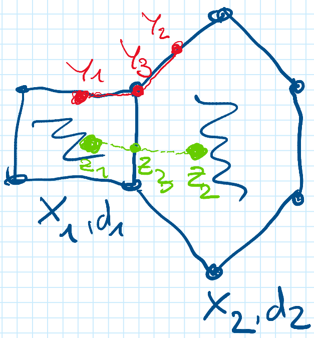
Q: $\boxed{\text{Does this } d = (d_s)_{s\in S} \text{ matter?}}$

→ "No": Two different choices yield homeomorphic metrics on the Coxeter Polytopes with the same angles!

Q: How do we get a metric on the entire complex?

A: §2 Polyhedral complexes + gluing metrics.

Example:



We want: $d|_{X_i} = d_i \quad i = 1,2$.

$+ \; d(y_1,y_2) = d_1(y_1,y_3) + d_2(y_3,y_2)$

$d(z_1,z_2) = d_1(z_1,z_3) + d_2(z_3,z_2)$

→ given $X = X_1 \cup X_2$, $X_1 \cap X_2 \neq \emptyset$, $d_1|_{X_1\cap X_2} = d_2|_{X_1 \cap X_2}$

set $$d(x,y) = \begin{cases} \boxed{d_i(x,y)} & \boxed{x,y \in X_i} \\ \inf\limits_{\boxed{z\in X_1 \cap X_2}} d(x,z) + d(z,y) & \end{cases}$$

→ generalize this idea for cell complexes.

⌜Not difficult, just a bit more technical...⌟

Prop: Given a cell complex where each cell is geodesic and which is

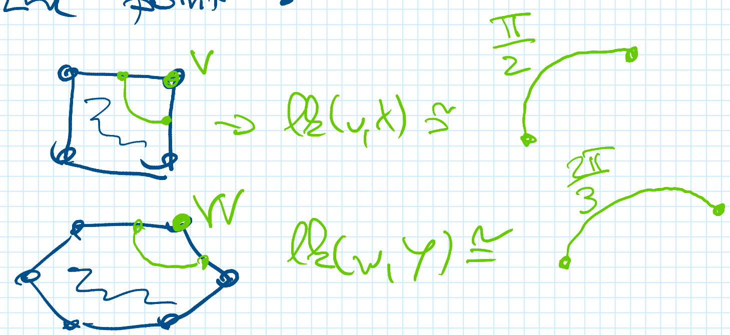
Prop: Given a cell complex where each cell is geodesic and which is locally finite, we obtain a geodesic metric using the "gluing metric construction".

→ From now on assume that Σ is equipped with this metric for some choice of d.

§3 Links

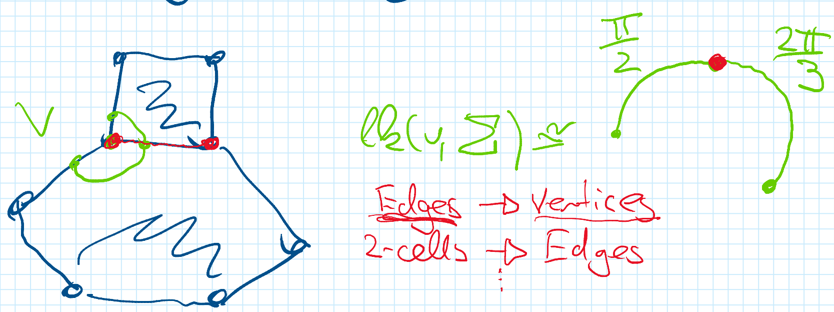
→ We need to study links to check whether Σ is CAT(0).

→ Given a vertex $v \in \Sigma$, the link of v is "the space of directions pointing into Σ". The link in each cell has a natural metric given by the 'angles in the point':



Each vertex $v \in \Sigma$ is part of finitely many cells, so we can metrize $\mathrm{lk}(v,\Sigma)$ using the "gluing metric":

Ex:



§4 Checking the link Condition

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Case 1: Right-angled case:

$\mathrm{ord}(st) \in \{2, \infty\} \quad \forall s,t \in S.$

$\Rightarrow$ all Coxeter-Polytopes are cubes.

$\Rightarrow$ all edge lengths in $lk(v, \Sigma)$ for any vertex are precisely $\frac{\pi}{2}$!

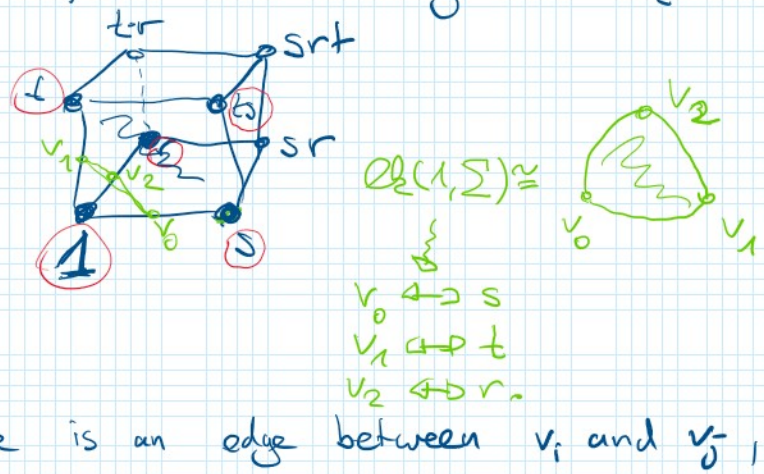
So, by Thm C(1), $lk(v, \Sigma)$ is CAT(1) $\Leftrightarrow$ it is flag.

$\Rightarrow$ It suffices to compute $lk(1, \Sigma)$ ($\to W_\emptyset$), since multiplication with $w \in W$ is an isometry.

$\Rightarrow$ Given $\{v_0, \dots, v_k\} \subseteq \mathrm{Vert}(lk(1, \Sigma))$ pairwise joined by edges, we need to show that $\{v_0, \dots, v_k\}$ spans a simplex in $lk(1, \Sigma)$.

$\Rightarrow$ Since the 1-skeleton of Σ is $\boxed{\mathrm{Cay}(W, S)}$, each vertex $v_i \in \{v_0, \dots, v_k\}$ corresponds to an edge from 1 to a generator $s_i \in S$.

Image:



$\Rightarrow$ Since there is an edge between v_i and v_j, $1, s_i, s_j$ lie in a 2-cell, so s_i and s_j commute.

$\Rightarrow$ Therefore $\{s_0, \dots, s_k\}$ (commute!) is a spherical subset of S. So $\{v_0, \dots, v_k\}$ spans a simplex in $lk(1, \Sigma)$.(*) Thus Σ is CAT(0). (*) $\Rightarrow$ flag!

Case 2: (W, S) is not necessarily right-angled.

$\Rightarrow \boxed{lk(1, \Sigma)} \cong L \leftarrow$ nerve of (W, S).

Case 2: (W,S) is not necessarily right-angled.

$\rightarrow$ $\boxed{Lk(1,\Sigma) \cong L}$ $\leftarrow$ nerve of (W,S).

$\rightsquigarrow$ if $\{v_0, \ldots, v_k\} \subseteq Vert(Lk(1,\Sigma))$ is pairwise connected by edges, then $\{s_0, \ldots, s_k\}$ is spherical $\Leftrightarrow$ $\boxed{\{u_{s_0}, \ldots, u_{s_k}\}}$ spans a spherical simplex (!!!)

$\rightarrow$ Moussong's Lemma (Thm C ②) shows Σ is CAT(0).

§5 Summary + Outlook

- $\boxed{\text{Thm:}}$ Coxeter groups are CAT(0) groups

$\rightarrow$ Many properties

↳ abelian subgroups are finitely generated.

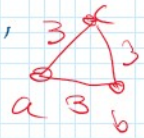
$\rightarrow$ finitely many conjugacy classes of finite subgroups

$\rightarrow$ ...

- We can also construct Σ "in hyperbolic space" and see, when Coxeter Groups are "δ-hyperbolic" ($\Leftrightarrow$ no $\mathbb{Z}^2$ subgroup!!!).

- "Flat torus Theorem": affine Coxeter Groups

If $\mathbb{Z}^n \curvearrowright X$ CAT(0) (semi-simple), then we can "say stuff about the action and space". For example:



$\rightarrow$ Σ regular tiling of $\mathbb{E}^2$ by triangles!

  - $Min(\mathbb{Z}^n) \cong \boxed{Y \times \mathbb{E}^n}$
  - $g \in Isom(X)$ with $g \cdot \mathbb{Z}^n g^{-1} = \mathbb{Z}^n$ then $g \cdot Min(\mathbb{Z}^n) = Min(\mathbb{Z}^n)$ and g preserves product structure.

  ⋮

- More Applications as a geometric model for a gp. is very useful!